The Fundamental Constants of Orthotropic Affine Slab/Plate Equations

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It was previously delineated by the author that the solution *forms* for specially orthotropic plates and slabs in a suitable affine space depend primarily on the constants D^* and H^* and that the rotational behavior of modulus and compliance matrices depend on a nondimensional constant $B = U_2/4U_3$ (where the U_i invariants are either modulus or compliance related). The mathematical structure of these constants is examined, revealing the following principal conclusions: 1) D^* is less than or equal to unity for *all physically possible orthotropic materials* (most orthotropic plate solutions in the literature correspond to $D^* > 1$); 2) rotationally anomalous behavior is found for numerous materials, including Kevlar 49/epoxy and B(4)5505 boron/epoxy (incorrect contrary statements are found in the literature); and 3) a simple inequality can be derived to identify regular or anomalous behavior. The latter is presented in two forms.

Nomenclature

A_{ij}, B_{ij}, D_{ij}	= plate constants
B	= anomaly constant for modulus components, see Eq. (13)
B_*	= anomaly constant for compliance components
D^*	= generalized rigidity, see Eq. (7)
$\mathfrak{D}$	= isotropic plate stiffness constant
E_{11}, E_{22}	= longitudinal and transverse engineering elastic constants
E_f, E_m	= longitudinal modulus of fiber and matrix, respectively
G_{12}	= engineering shear constant
G_f, G_m	= shear modulus of fiber and matrix, respectively
H^*	= generalized slab rigidity, see Eq. (8)
$\bar{Q}_{ij}$	= off-axis components of modulus
$\bar{S}_{ij}$	= off-axis components of compliance
U_i, U_i^*	= invariants of modulus and compliance, respectively
V_f, V_m	= volume (%/100) of fiber and matrix, respectively
w	= lateral plate displacement
x, y	= plate physical dimensions
x_0, y_0	= plate affine dimensions
ϵ	= generalized Poisson's ratio
θ	= off-axis fiber angle
ν_{12}	= principal Poisson's ratio
ν_f, ν_m	= Poissons's ratio of fiber and matrix, respectively
ϕ	= stress function
∇^4	$= \dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$, biharmonic operator

Foreword

THIS paper examines the elastic properties of orthotropic slabs/plates as a first step in a process that may eventually provide a method for understanding, in a simple fashion, the parametric role of the elastic constants in governing the response of composite laminates. Since most laminates are comprised of specially orthotropic lamina arranged in some (angle) stacking sequence, the idea appears promising.† The paper divides into two hierarchically related topics. The first part discusses the global constants associated with orthotropic slab/plate equations (D^*, H^*, and ϵ), while the second part discusses the rotational behavior of the modulus/compliance components ($\bar{Q}_{ij}$ and $\bar{S}_{ij}$) associated with orthotropic slab/plate materials.

I. The Global Relations

The Emergence of D^* and H^* as Fundamental Constants

The orthotropic operators (on ϕ and w, respectively)

$$\frac{1}{E_{22}}\frac{\partial^4 \phi}{\partial x^4} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right)\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{1}{E_{11}}\frac{\partial^4 \phi}{\partial y^4} \tag{1}$$

and

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} \tag{2}$$

which characterize slab/plate behavior, are the well-known counterparts to $\nabla^4\phi$ and $\mathfrak{D}\nabla^4 w$. A principal difficulty in discerning analysis trends of the equations related to the above operators arises because of the four elastic constants present in each operator. It has been shown previously that the respective affine transformations[2-6]

$$x = (E_{11})^{1/4} x_0, \quad y = (E_{22})^{1/4} y_0 \tag{3}$$

and

$$x = (D_{11})^{1/4} x_0, \quad y = (D_{22})^{1/4} y_0 \tag{4}$$

allow the operators of Eqs. (1) and (2) to be expressed as‡

†In completed but as yet unpublished work, the promise has been fulfilled for generally laminated plates. A specific result for regular symmetric angle-ply plates[1] was presented at the 55th Shock and Vibration Symposium in 1984.

‡The plate operator has been used in Refs. 4-6 to investigate buckling, vibration, and similarity rules of specially orthotropic material and these ideas have been further applied in a number of theses written at both the Rensselaer Polytechnic Institute and the U.S. Air Force Institute of Technology by former students of the author.

Received May 1, 1984; revision received March 10, 1985.

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$$\frac{\partial^4 \phi}{\partial x_0^4} + 2H^* \frac{\partial^4 \phi}{\partial x_0^2 \partial y_0^2} + \frac{\partial^4 \phi}{\partial y_0^4} \quad (5)$$

and

$$\frac{\partial^4 w}{\partial x_0^4} + 2D^* \frac{\partial^4 w}{\partial x_0^2 \partial y_0^2} + \frac{\partial^4 w}{\partial y_0^4} \quad (6)$$

where

$$D^* = (D_{12} + 2D_{66})/(D_{11}D_{22})^{1/2} \quad (7)$$

and

$$H^* = \left(\frac{1}{2G_{12}} - \frac{\nu_{12}}{E_{11}}\right)(E_{11}E_{22})^{1/2} \quad (8)$$

By defining $\epsilon D^* = D_{12}/(D_{11}D_{22})^{1/2}$, it can be shown that D^*, H^* and ϵ are interrelated by the expression

$$(H^*)^{-1} = D^*(1-\epsilon)/[1-\epsilon(D^*)^2] \quad (9a)$$

or by an alternate form

$$H^*(1-\epsilon)/\epsilon^{1/2} = -\epsilon^{1/2}D^* + (1/\epsilon^{1/2}D^*) \quad (9b)$$

Clearly, H^* and D^* are of fundamental importance to the above affine plate operators. (It turns out that solutions depend weakly on ϵ; thus, *either* H^* or D^* can be considered as the fundamental constant—the current choice being D^*.) A useful bit of information would be, what are the values of D^* for existing orthotropic materials? The results of this question are presented and discussed in the next subsection; they are interesting enough to prompt the formulation of a *tentative law*. For completeness, a quantity associated with D^* for off-axis angles is presented and discussed in the Appendix.

Values of D^* for Existing Materials

In a cursory review of literature,[7-12] 20 orthotropic materials were found with listed elastic constants (usually in the form E_{11}, E_{22}, G_{12}, and ν_{12} instead of zero-angle compliance and modulus terms). The associated D^*, ϵ, H^* [and also $(H^*)^{-1}$], and B (for both modulus and compliance) are presented in Table 1, in which the materials have been arranged in order of descending values of D^*. [Anomaly constant B (for both modulus and compliance) is discussed in Sec. II.] Examination of Table 1 reveals the following trends:

1) D^* is less than unity. In fact, $D^* = 1.0$ corresponds to Huber's "suggested standard form" and would be the isotropic case if, additionally, $D_{11}/D_{22} = 1$. Huber's results are called quasi-isotropic by Brunelle and Oyibo.[5]

2) The generalized Poisson's ratio ϵ has a range of 0.14-0.63; but, if only the common, modern materials are examined (such as GR/E and "Kevlar-like" materials), the ϵ range is reduced to 0.15-0.30.

3) H^* is greater than unity.

Thus, a *tentative* law and a corollary law would be:

The range of the fundamental plate constant D^* is the closed interval zero to one.
The range of the inverse plate constant $(H^*)^{-1}$ is the closed interval zero to one.

The word tentative has been emphasized; the next subsection provides a more satisfying basis for re-enunciating these laws.

Values of D^* Calculated by Micromechanics Theory

By using the simplest micromechanical relations,[13]

$$\frac{E_{11}}{E_m} = V_f\left(\frac{E_f}{E_m}\right) + V_m, \quad \frac{E_m}{E_{22}} = \frac{V_f}{(E_f/E_m)} + V_m$$

$$\frac{E_m}{G_{12}} = \frac{V_f}{(G_f/G_m)} + V_m, \quad \nu_{12} = \nu_f V_f + \nu_m V_m$$

where

$$V_f + V_m = 1$$

D^* may be expressed as a function of V_f, E_f/E_m, and ν_f and ν_m. Using intermediate functions A_0, and B_0, defined as

$$A_0 = \nu_{12}(E_{22}/E_{11})^{1/2}$$

and

$$B_0 = G_{12}/(E_{11}E_{22})^{1/2}$$

Table 1 Fundamental constants for 20 materials

BM[a]	BC[b,c]	H^*	$(H^*)^{-1}$	D^*	ϵ	E_{11}	E_{22}	G_{12}	ν_{12}	$(E_{11}^*E_{22})^{1/2}$
1.368	−4.941	1.03	0.971	0.715	0.186	1.400	0.117	0.120	0.460	0.405
1.297	1.333	1.01	0.993	0.710	0.203	7.800	2.600	1.300	0.250	4.503
1.155	1.165	1.28	0.783	0.577	0.209	5.600	1.200	0.600	0.260	2.592
1.157	6.676	1.79	0.559	0.504	0.080	8.800	0.734	0.590	0.140	2.541
0.972	0.514	1.33	0.754	0.504	0.281	6.200	1.700	0.600	0.270	3.247
1.139	5.913	1.69	0.591	0.478	0.160	20.000	1.300	1.030	0.300	5.099
1.120	2.438	1.75	0.570	0.460	0.164	13.500	1.140	0.758	0.260	3.923
1.102	2.748	1.90	0.527	0.425	0.169	22.900	1.500	1.040	0.280	5.861
0.130	0.025	1.45	0.688	0.380	0.413	26.000	22.100	2.740	0.170	23.971
0.747	0.282	2.18	0.460	0.365	0.187	12.900	4.150	1.090	0.120	7.317
1.062	67.155	2.51	0.399	0.339	0.137	42.000	1.000	0.950	0.300	6.481
1.004	0.677	2.36	0.424	0.329	0.207	20.000	2.100	0.850	0.210	6.481
1.027	0.732	2.19	0.457	0.315	0.291	11.000	0.798	0.334	0.340	2.963
0.991	0.489	2.20	0.455	0.304	0.312	30.000	3.000	1.000	0.300	9.487
1.022	0.761	2.43	0.412	0.297	0.260	21.000	1.400	0.600	0.300	5.422
0.995	0.610	3.01	0.332	0.257	0.217	30.000	2.100	0.800	0.210	7.937
0.967	0.414	2.84	0.352	0.250	0.278	29.600	2.700	0.811	0.230	8.940
0.418	0.071	3.53	0.283	0.215	0.234	1.200	0.600	0.070	0.071	0.849
0.792	0.081	1.68	0.594	0.212	0.632	18.000	3.600	0.320	0.300	8.050
1.013	1.013	4.02	0.249	0.197	0.200	30.000	0.750	0.375	0.250	4.743

[a]Value of B related to modulus. [b]Value of B related to compliance. [c]Obviously two values of BC look suspicious, i.e., (−4.941 and 67.155); to date the accuracy of the raw data for these values is unconfirmed.

D^* is given as

$$D^* = A_0 + 2B_0(1 - A_0^2) \tag{10}$$

This relation is plotted in Fig. 1, where it is immediately noticed that the range of D^* is bounded by zero and unity. This result, in conjunction with the results of Eq. (9a) as displayed in Fig. 2, forms a *strong basis* for re-enunciating the D^* and H^* *range laws.* Thus,

The range of the fundamental plate/slab constants D^* and H^*-inverse is from zero to one.

If the law is valid, then a firm bound exists on the possible values of D^* (and H^*) for all materials (*even those not yet in existence*!); quoting from Brunelle and Oyibo[5]: "Of the three possible cases discussed in the specially orthotropic literature, i.e., case 1: $D^* < 1$; case 2: $D^* = 1$; case 3: $D^* > 1$; usually *only* case 3 has been used in presenting numerical results. Thus, if the above law is valid (i.e., $0 \le D^* \le 1$), a corollary result of importance is that many numerical results in the orthotropic plate literature are physically incorrect." Obviously, it is predicted that the slab solutions for $H^* \ge 1$ *have correct* numerical results, that is, they correspond to physical reality.

II. The Rotational Relations

Ignoring the simple rotational behavior of $\bar{Q}_{12}$ and $\bar{Q}_{66}$ (i.e., $\sim\cos 4\theta$) and noting that $\bar{Q}_{22}$ and $\bar{Q}_{26}$ are related to $\bar{Q}_{11}$ and $\bar{Q}_{16}$, respectively, by simple $+/-$ phase relations, only $\bar{Q}_{11}$ and $\bar{Q}_{16}$ need be examined. These can be expressed as

$$\bar{Q}_{11} = U_1 + U_2\cos 2\theta + U_3\cos 4\theta \tag{11}$$

and

$$\bar{Q}_{16} = \tfrac{1}{2} U_2 \sin 2\theta + U_3 \sin 4\theta \tag{12}$$

(See Ref. 13 for an excellent overview of basic modulus and compliance concepts.)

Defining $\bar{Q}_{11}^* = (\bar{Q}_{11} - U_1)/U_3$, $\bar{Q}_{16}^* = \bar{Q}_{16}/U_3$, and B as

$$B = U_2/4U_3 \tag{13}$$

it is noted that parameter B completely controls the rotational behavior. Thus,

$$\bar{Q}_{11}^* = 4B\cos 2\theta + \cos 4\theta \tag{14}$$

$$\bar{Q}_{16}^* = 2B\sin 2\theta + \sin 4\theta \tag{15}$$

Obviously, if B is "small enough," the 4θ behavior will dominate and vice versa; closer scrutiny will show $B = 1$ to be the boundary between "normal 2θ-like behavior" and a more interesting behavior. Toward these ends the conditions for the local extrema of $\bar{Q}_{11}^*$ and $\bar{Q}_{16}^*$ are summarized, using the identities $\sin 4\theta = 2\sin 2\theta \cos 2\theta$ and $\cos 4\theta = 2\cos^2 2\theta - 1$,

$\dfrac{\partial \bar{Q}_{11}^*}{\partial \theta} = 0$ yields:

$$\theta = n\pi/2; \quad n = 0,1,2,\ldots \tag{16a}$$

$$\theta = \tfrac{1}{2}\arccos(-B); \quad |B| \le 1 \tag{16b}$$

and

$\dfrac{\partial \bar{Q}_{16}^*}{\partial \theta} = 0$ yields:

$$\cos 2\theta = -\frac{B}{4} \pm \sqrt{\left(\frac{B}{4}\right)^2 + \frac{1}{2}} \tag{17}$$

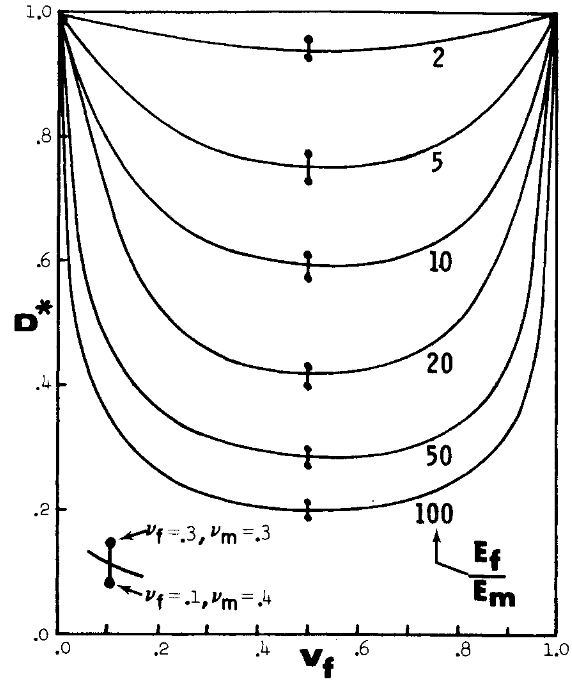

Fig. 1 D^* vs V_f with E_f/E_m as a parameter.

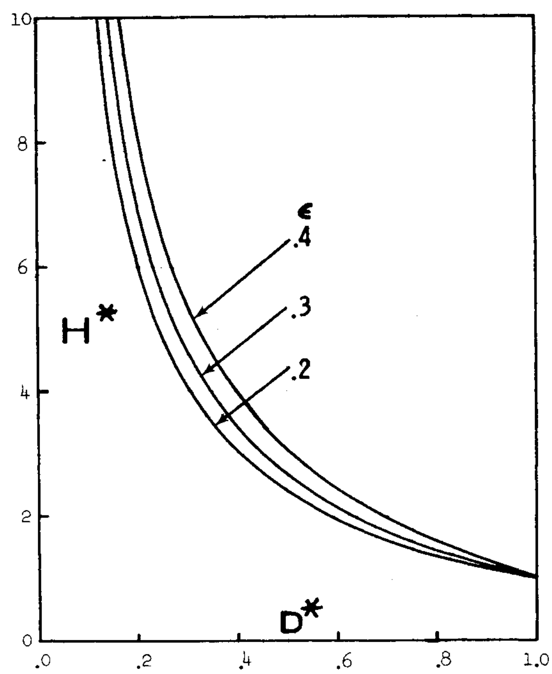

Fig. 2 H^* vs D^* with ϵ as a parameter.

Note particularly that, when $B < 1$, a second set of extrema appears for $\bar{Q}_{11}^*$ and that, when $B \le 1$ (a slightly different inequality), a second set of extrema appears for $\bar{Q}_{16}^*$. The rotational behavior for $B < 1$ is called *anomalous* and thus B is called the anomaly constant. Referring to Table 1 (the BM column), it is seen that 8 of the 20 materials have modulus-related values of *B less than unity*. Therefore, in these cases second sets of extrema will appear in the $\bar{Q}_{11}^*$ and $\bar{Q}_{16}^*$ vs θ curves. Figure 3 displays a pair of such curves for $B = 0.8$, where it is observed that:

1) $\bar{Q}_{11}^*$ has a local *maximum* at $\theta = 90$ deg and local minimums at θ values 71.6 and 108.4 deg.

2) $\bar{Q}_{16}^*$ has a local *minimum* at $\theta = 79.6$ deg (and a local maximum at $\theta = 100.4$ deg), so that a region exists about $\theta = 90$ deg in which the sign of $\bar{Q}_{16}$ (and also $\bar{Q}_{26}$) is opposite to what is normally expected.

These observations may play an important role in design/off-design calculations for such disciplines as aeroelastic tailoring that make extensive use of the $\bar{Q}_{16}$, $\bar{Q}_{26}$ terms.

With a little manipulation, the (modulus related) inequality for $B<1$ can be expressed as§

$$\frac{G_{12}}{E_{22}} < \frac{(1-\nu_{12})/2}{1-\nu_{12}^2(E_{22}/E_{11})} \tag{18}$$

The graph of Eq. (18) is shown in Fig. 4. This graph and its compliance-related counterpart is a convenient "map" for observing the "B trend" of new materials (i.e., $B>1$ or $B<1$). Another expression for $B<1$ in terms of Q^* and E_{22}/E_{11} is given by

$$Q^* < (E_{22}/E_{11})^{1/2} \tag{19}$$

where Q^* is given by¶

$$Q^* = (Q_{12} + 2Q_{66})/(Q_{11}Q_{22})^{1/2} \tag{20}$$

From what has been said, it is clear that the rotational behavior of the compliances $\bar{S}_{ij}$ can be similarly examined. The corresponding anomaly constant for the compliance is denoted by B_* and is expressed as

$$B_* = \frac{U_2^*}{4U_3^*} = \frac{(E_{22}/E_{11}) - 1}{(E_{22}/E_{11}) - (E_{22}/G_{12}) + 2\nu_{12}(E_{22}/E_{11}) + 1} \tag{21}$$

Referring to Table 1, it is seen that 12 of the 20 materials have values of B_* (see the BC column) less than unity. Hence, the anomalous effects are more prevalent (and pronounced) for the compliance quantities $\bar{S}_{ij}$.

Appendix: D^* for an Anisotropic Lamina

Among several quantities associated with D^* for off-axis directions (they are all related by angle dependent scale factors), a convenient choice is $\bar{D}^*$ defined as

$$\bar{D}^* = (\bar{D}_{12} + 2\bar{D}_{66})/\sqrt{D_{11}D_{22}} \tag{A1}$$

In particular, note that the denominator contains the principal values of D_{11} and D_{22}, i.e., they are *not* barred quantities.

Given the lamina definitions for $\bar{D}_{12}$ and $\bar{D}_{66}$, namely

$$\bar{D}_{12} = (D_{11} + D_{22} - 4D_{66})\sin^2\theta\cos^2\theta + D_{12}(\sin^4\theta + \cos^4\theta) \tag{A2}$$

and

$$\bar{D}_{66} = (D_{11} + D_{22} - 2D_{12} - 2D_{66})\sin^2\theta\cos^2\theta + D_{66}(\sin^4\theta + \cos^4\theta) \tag{A3}$$

some manipulation using trigonometric identities reveals a particularly simple form for $\bar{D}^*$. It is expressed as

$$\bar{D}^* = D^* + 3/4(K^{-1} + K - 2D^*)\sin^2 2\theta \tag{A4}$$

where

$$D^* = (D_{12} + 2D_{66})/\sqrt{D_{11}D_{22}} \tag{A5}$$

§For large enough E_{11}/E_{22}, Eq. (18) has the lower bound given by $G_{12}/E_{22} < (1-\nu_{12})/2$.

¶For a single layer, $D^* = (t^3/12)Q^*$.

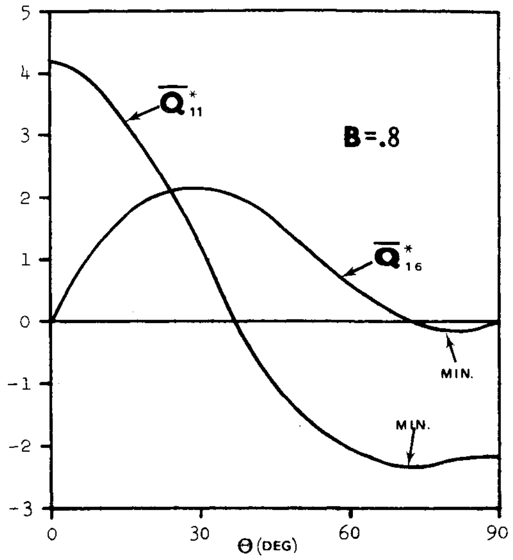

Fig. 3 $\bar{Q}_{11}^*$ and $\bar{Q}_{16}^*$ vs θ.

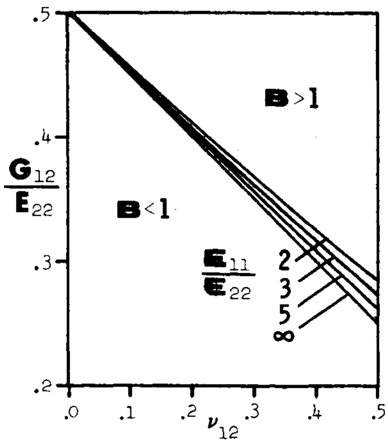

Fig. 4 Anomaly constant map in the $G_{12}/E_{22} - \nu_{12}$ plane.

and

$$K = \sqrt{D_{22}/D_{11}} \tag{A6}$$

The difference $\Delta\bar{D}^* \equiv \bar{D}^* - D^*$ depends only on a K and D^* dependent coefficient [namely 0.75 $(K^{-1} + K - 2D^*)$] and a simple trigonometric function, $\sin^2 2\theta$. The angles for maximum $\Delta\bar{D}^*$ and $\bar{D}^*$ are $\pm(\pi/4)$ and $\pm(3\pi/4)$. The dependence of $\Delta\bar{D}^*$ on D^* and K is obvious.

However, the most important feature of Eq. (A4) is its overall statement,

$$\bar{D}^* \geq D^* \tag{A7}$$

which does not in any way infringe on the statement that

$$0 \leq D^* \leq 1 \tag{A8}$$

Indeed the above statements are uncoupled from one another in the sense that, although $\bar{D}^*$ depends on D^*, it has no influence on the *range* of D^* that has been established in the body of the text.

Acknowledgments

This research was carried out at the Rensselaer Polytechnic Institute under a joint NASA/AFOSR Grant NGL33-018-033; these agencies are thanked for their generous support and encouragement. It is also a pleasure to thank Ms. Amy Moore for her competent typing skills.

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